LECTURE 19

Math 125

HW 6, due Friday, 7.4/1, 2b, 3b, 8, 9, 17, 21, 26, 28, 31, 39, 42
7.5, 7.7
Parts 4 & 5 of Week 6 Sheet

7.7 Approximating Definite Integrals

Sometimes we cannot find an antiderivative no matter how hard we try

Ex
$$\int_0^1 e^{-x^2}\,dx \qquad \int_0^2 \sqrt{1+x^5}\,dx$$
$$\int_3^8 \sin(x^3)\,dx$$

Goal: Learn how to compute accurate approximations.

We already have 3 methods to do this

L_n = left endpoint rectangles
R_n = right endpoint rectangles
M_n = midpoint rectangles

Recall: To approximate $\int_a^b f(x)\,dx$

I Break up into n subdivisions (width of each $= \Delta x = \frac{b-a}{n}$)

II $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, ..., $x_n = b$

$$L_n = \Delta x\, f(x_0) + \Delta x\, f(x_1) + \cdots + \Delta x\, f(x_{n-1})$$

$$\boxed{\begin{aligned} &1 \quad L_n = \Delta x\,[f(x_0) + f(x_1) + \cdots + f(x_{n-1})] \\ &2 \quad R_n = \Delta x\,[f(x_1) + f(x_2) + \cdots + f(x_n)] \\ &3 \quad M_n = \Delta x\,[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)] \\ &\qquad \bar{x}_i = \tfrac{1}{2}(x_{i-1} + x_i) \end{aligned}}$$

Example $y=\sqrt{100-x^3}$ $x=0$ to $x=3$

Approximate $\int_0^3 \sqrt{100-x^3}\,dx$ using $n=3$ subdivisions.
(L_3, R_3, M_3)

I $\Delta x = \frac{3-0}{3} = 1$

II $x_0=0$, $x_1=1$, $x_2=2$, $x_3=3$

$\boxed{\text{SHOW OVERHEAD}}$

$L_3 = \Delta x\,[f(0)+f(1)+f(2)]$
$= 1\,[\sqrt{100-0^3}+\sqrt{100-1^3}+\sqrt{100-2^3}] \approx \boxed{29.541537}$

$R_3 = \Delta x\,[f(1)+f(2)+f(3)]$
$= 1\,[\sqrt{100-1^3}+\sqrt{100-2^3}+\sqrt{100-3^3}] = \boxed{28.085541}$

$M_3 = \Delta x\,[f(\frac{1}{2})+f(\frac{3}{2})+f(\frac{5}{2})]$
$= 1\,[\sqrt{100-(\frac{1}{2})^3}+\sqrt{100-(\frac{3}{2})^2}+\sqrt{100-(\frac{5}{2})^3}] = \boxed{29.009136}$

Today we will discuss 2 more methods

(4) $\boxed{\text{Trapezoid Rule}}$

$T_n = \frac{1}{2}\Delta x\,[f(x_0)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)]$

(5) $\boxed{\text{Simpson's Rule}}$ (n must be even)

$S_n = \frac{1}{3}\Delta x\,[f(x_0)+4f(x_1)+2f(x_2)$
$+4f(x_3)+2f(x_4)$
$+\cdots$
$+2f(x_{n-2})+4f(x_{n-1})+f(x_n)]$

(Alternate 4's & 2's)

For $\int_0^3 \sqrt{100-x^3}\,dx$

$T_3 = \frac{1}{2}\Delta x\,[f(0)+2f(1)+2f(2)+f(3)]$
$= \frac{1}{2}\cdot 1\,[\sqrt{100-0^3}+2\sqrt{100-1^3}+2\sqrt{100-2^3}+\sqrt{100-3^3}] = \boxed{28.81354}$

$S_6 = \frac{1}{3}\underset{\frac{3}{6}}{\underset{\uparrow}{\Delta x}}\,[f(0)+4f(\frac{1}{2})+2f(1)+4f(\frac{3}{2})+2f(2)+4f(\frac{5}{2})+f(3)]$

$= \boxed{28.944105}$

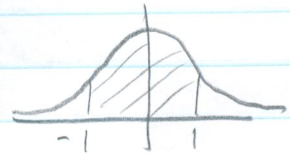
$\boxed{\text{ACTUAL VALUE}}$
28.94418784

Note: • Midpoint & Trapezoid about equally good.
• Simpson's better

Now we will see where these methods come from by looking at a special example

Consider $y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

Aside: The graph looks like which is the "bell curve"



The total area from $-\infty$ to ∞ is 1 (or 100%)

The integral $\boxed{\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx}$ = area within one standard deviation

≈ 0.68

We will try to better approximate this value.

If $n=4$, what is Δx, x_0, x_1, x_2, x_3, x_4?

I $\Delta x = \frac{1 - -1}{4} = \frac{1}{2}$

II $x_0 = -1$, $x_1 = -\frac{1}{2}$, $x_2 = 0$, $x_3 = \frac{1}{2}$, $x_4 = 1$

① $L_4 = \Delta x[f(-1) + f(-\frac{1}{2}) + f(0) + f(\frac{1}{2})] = 0.6725$

② $R_4 = \Delta x[f(-\frac{1}{2}) + f(0) + f(\frac{1}{2}) + f(1)] = 0.6725$

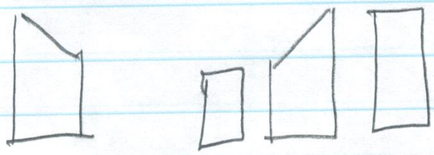
you do

③ $M_4 = \Delta x[f(-\frac{3}{4}) + f(-\frac{1}{4}) + f(\frac{1}{4}) + f(\frac{3}{4})] = 0.6878$

Trapezoid rule

Idea: instead of rectangles use trapezoids

Overhead



Note: Each trapezoid has area halfway between the left & right endpoint rectangles

So $T_n = \frac{1}{2}(L_n + R_n)$ = average of left and right endpoint method

$$L_4 = \Delta x\,[f(x_0) + f(x_1) + f(x_2) + f(x_3)]$$
$$+ \quad R_4 = \Delta x\,[f(x_1) + f(x_2) + f(x_3) + f(x_4)]$$

$$L_4 + R_4 = \Delta x\,[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4))$$

$$T_4 = \frac{1}{2}\Delta x\,[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$$

$$\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx \approx T_4$$

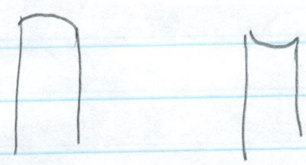
$$= \frac{1}{2}\left(\frac{1}{2}\right)[f(-1) + 2f(-\tfrac{1}{2}) + 2f(0) + 2f(\tfrac{1}{2}) + f(1)]$$
$$= \frac{1}{4}\left[\frac{1}{\sqrt{2\pi}} e^{-\frac{(-1)^2}{2}} + 2\frac{1}{\sqrt{2\pi}} e^{-\frac{(-\frac{1}{2})^2}{2}} \cdots\right.$$
$$= \boxed{0.672518}$$

Good for a graph that doesn't have much curve.

Note: · L_n, R_n, T_n are the same? because of symmetry
· M_n is the best so far

Simpson's Rule

Idea Instead of using rectangles or trapezoids (i.e. top = a line)
use an object where the top is curved

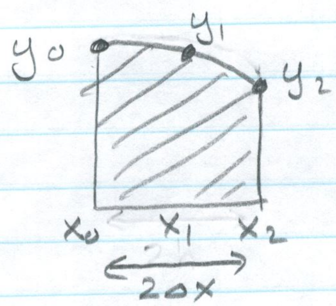


For simpson's rule we use a parabola.

I. Break into subintervals
II. put a parabola on top of each
III. Compute the area

Overhead

A



$$\text{AREA} = \frac{\Delta x}{3}(y_0 + 4y_1 + y_2)$$

Note: 3 pts get used at once

B

$$\int_a^b f(x)dx \approx S_4 = \frac{\Delta x}{3}(y_0 + 4y_1 + y_2) + \frac{\Delta x}{3}(y_2 + 4y_3 + y_4)$$

$$= \frac{1}{3}\Delta x[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4]$$

$$= \frac{1}{3}\Delta x[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4))$$

$$S_4 = \frac{1}{3}\frac{1}{2}\left[f(-1) + 4f(-\tfrac{1}{2}) + 2f(0) + 4f(\tfrac{1}{2}) + f(1)\right]$$
$$= 0.6827109757$$

TABLE

Actual value = 0.6826894921

Summary

$M_n = \Delta x\left[f(\bar{x}_1) \quad f(\bar{x}_n)\right]$

$T_n = \frac{1}{2}\Delta x\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$

$S_n = \frac{1}{3}\Delta x\left[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$

Error Bounds

Trapezoid

Suppose $|f''(x)| \le K$ for $a \le x \le b$

then Error for trapezoid $= \left|\int_a^b f(x)dx - T_n\right|$
$\le \frac{K(b-a)^3}{12n^2}$

Simpsons

Suppose $|f^{(4)}(x)| \le k$ for $a \le x \le b$
then $\left|\int_a^b f(x)dx - S_n\right| \le \frac{k(b-a)^5}{180n^4}$

Give an n value that gaurantees that T_n and S_n within 0.0001 of $\int_1^2 \frac{1}{x^2}dx$

$a=1, b=2$ $\quad f(x) = \frac{1}{x^2} \Rightarrow f'(x) = -\frac{2}{x^3}$

$f''(x) = \frac{6}{x^4}$ $\quad |f''(x)| = \left|\frac{6}{x^4}\right| \le 6$

WANT $\frac{6(2-1)^3}{12n^2} \le 0.0001$ $\quad f'''(x) = -\frac{24}{x^5}$

$5000 \le n^2$ $\quad f^{(4)}(x) = \frac{120}{x^6}$ $\quad |f^{(4)}(x)| = \left|\frac{120}{x^6}\right| \le 120$

$70.7106 \le n$

$\frac{120(2-1)^5}{180n^4} \le 0.0001$

$9.03 \le n$